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On curvature homogeneous three-dimensional Lorentzian manifolds

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Abstract

The aim of this paper is the study of three-dimensional Lorentzian manifolds whose Ricci tensor has three equal constant eigenvalues, whose associated eigenspace is two-dimensional. A complete local classification of this class of curvature homogeneous manifolds is presented. It turns out that, if the eigenvalue is zero, these are exactly the curvature homogeneous manifolds modelled on an indecomposable, non-irreducible Lorentzian symmetric space, which were first studied in Cahen et al. (1990), and the techniques presented in this paper can therefore be applied to obtain a complete (local) classification of these manifolds, and to construct a number of new examples of such manifolds.

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1. Introduction

A pseudo-Riemannian manifold (M, g) is said to be *curvature homogeneous* [23] if, for every pair of points $p, q \in M$, there exists a linear isometry $F : T_pM \to T_qM$ such that $F^*R_q = R_p$. Every locally homogeneous pseudo-Riemannian manifold is curvature homogeneous, and a curvature homogeneous space (M, g) is said to have *the same curvature tensor* as a homogeneous space $(\bar{M}, \bar{g})$ if, for any pair of points $m \in M$ and $\bar{m} \in \bar{M}$ there exists a linear isometry $F : T_mM \to T_{\bar{m}}\bar{M}$ such that $F^*\bar{R}_{\bar{m}} = R_m$. In this case $(\bar{M}, \bar{g})$ is said to be a (*homogeneous*) *model space* for (M, g).

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In [23], Singer states the problem of constructing non-homogeneous curvature homogeneous manifolds. This problem was extensively studied by many authors (see, e.g., [13–15,21,22,24]) and a large number of such examples were discovered. For a survey of known results and for more detailed information on the subject of curvature homogeneity and related problems, we refer to [1,3,26]. The first examples of non-homogeneous curvature homogeneous Lorentzian spaces were found in [6], where the authors made a study of curvature homogeneous manifolds modelled on Lorentzian symmetric spaces and constructed a family of such manifolds parametrized by one function of one variable. In [20], Patrangenaru later obtained another family of curvature homogeneous Lorentzian manifolds, using a construction similar to the one in [15].

In a number of recent publications (e.g., [4,5,8–12,16–18]), the problem of (locally) classifying three-dimensional curvature homogeneous pseudo-Riemannian manifolds received considerable attention. It is well known that the Riemann curvature tensor of a three-dimensional Riemannian manifold is completely determined by its Ricci tensor. As a consequence, a three-dimensional Riemannian manifold is curvature homogeneous if and only if it has constant principal Ricci curvatures. If all principal Ricci curvatures are equal (and constant), the manifold is of constant curvature. In [8,12] the authors obtained a complete local classification of three-dimensional Riemannian manifolds with constant principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3$. In [4], an alternative proof for this classification result was obtained by using a technique introduced in [17]. Finally, in [10,11,16], the authors studied Riemannian three-manifolds with three distinct principal Ricci curvatures, and in [16] a complete classification of the manifolds of this type was obtained.

As in the Riemannian case, the Riemann curvature tensor of a Lorentzian three-dimensional manifold is completely determined by its Ricci tensor. Contrary to the Riemannian case however, the Ricci operator, i.e., the self-adjoint operator associated to the Ricci tensor, cannot always be diagonalized, although it can always be written in one of the following standard forms with respect to a pseudo-orthonormal basis $\{E_1, E_2, E_3\}$, where E_3 is a time-like unit vector (see e.g. [7,19]):

$$\begin{aligned} &\text{I:}\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \text{III:}\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \\ &\text{II:}\ \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}, \qquad \text{IV:}\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & -1 & b \pm 2 \end{pmatrix}. \end{aligned} \tag{1}$$

As a consequence, a three-dimensional Lorentzian manifold is curvature homogeneous if and only if its Ricci operator takes one of the forms given by (1), where a, b and c are constant along the manifold M, and a systematic investigation of curvature homogeneity should be made in all these cases.

As in the Riemannian case, if the Ricci tensor is diagonalizable with three equal (and constant) eigenvalues, the manifold is of constant curvature and hence locally homogeneous. The first non-trivial case is therefore that of a diagonalizable Ricci operator with constant eigenvalues $\rho_1 = \rho_2 \neq \rho_3$, and a complete local classification of the manifolds of this type was obtained in [5], where the present author corrected and extended the results given in [18].

In the present paper, we continue the study of the classification problem for curvature homogeneous three-dimensional Lorentzian manifolds. In particular, we investigate three-dimensional Lorentzian manifolds whose Ricci tensor has three equal constant eigenvalues, whose associated eigenspace is two-dimensional, i.e., the Ricci tensor is of type IV with $a = b \pm 1$. We will prove the existence of a family of such metrics, and determine which of these metrics are locally isometric, thereby obtaining a classification result similar to those presented in [4,5]. It turns out that, if the triple eigenvalue of the Ricci tensor is equal to zero, the manifolds considered here are exactly the Lorentzian manifolds modelled on an indecomposable, non-irreducible Lorentzian symmetric space studied in [6]. Our techniques therefore allow us to classify, at least locally, the Lorentzian manifolds of this type, and to obtain a number of new (non-homogeneous) examples of such manifolds.

The rest of this paper is organized as follows. In Section 2, we compute the necessary and sufficient conditions for a Lorentzian manifold to have a Ricci curvature tensor with three equal constant eigenvalues, whose associated eigenspace is two-dimensional. In Section 3, we provide a simple criterion to determine if two such manifolds are locally isometric. Finally, in Section 4, we study the differential equations of Section 2 in more detail. We investigate the local existence of the three-dimensional Lorentzian manifolds under consideration, and we determine which of these manifolds are locally isometric, thereby obtaining a classification result similar to those given in [4,5].

2. The basic differential equations

Let (M, g) be a three-dimensional Lorentzian manifold whose Ricci tensor has three equal constant eigenvalues, whose associated eigenspace is two-dimensional. Then, at least locally, there exists a pseudo-orthonormal frame field $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ such that (see, e.g., [7,19])

$$\begin{aligned} &g(\tilde{E}_1, \tilde{E}_1) = g(\tilde{E}_2, \tilde{E}_2) = 1, \qquad g(\tilde{E}_3, \tilde{E}_3) = -1, \\ &g(\tilde{E}_i, \tilde{E}_j) = 0 \quad \text{if } i \neq j \end{aligned} \tag{2}$$

and such that

$$\begin{aligned} &\rho(\tilde{E}_1, \tilde{E}_1) = \beta + \eta, \qquad \rho(\tilde{E}_2, \tilde{E}_2) = \beta, \\ &\rho(\tilde{E}_2, \tilde{E}_3) = 1, \qquad \rho(\tilde{E}_3, \tilde{E}_3) = -(\beta + 2\eta), \end{aligned} \tag{3}$$

where $\eta = \pm 1$, and where the remaining components of the Ricci tensor vanish. Replacing this pseudo-orthonormal frame field by

$$E_1 = \tilde{E}_1, \qquad E_2 = \frac{\tilde{E}_2 + \eta \tilde{E}_3}{\sqrt{2}}, \qquad E_3 = \frac{\tilde{E}_2 - \eta \tilde{E}_3}{\sqrt{2}}, \tag{4}$$

conditions (2) and (3) are equivalent to the (local) existence of a so-called "null" frame field $\{E_1, E_2, E_3\}$ such that

$$g(E_1, E_1) = g(E_2, E_3) = 1, \qquad g(E_i, E_j) = 0 \quad \text{otherwise}, \tag{5}$$

and

$$\begin{aligned} &\rho(E_1, E_1) = \rho(E_2, E_3) = 2\kappa, \qquad \rho(E_3, E_3) = -2\eta, \\ &\rho(E_i, E_j) = 0 \quad \text{otherwise}, \end{aligned} \tag{6}$$

where $2\kappa = \beta + \eta$.

The components of the Levi Civita connection ∇ can be written, with respect to this null frame field $\{E_1, E_2, E_3\}$, as

$$\begin{aligned} &\nabla_{E_1} E_1 = AE_2 + BE_3, &&\nabla_{E_2} E_1 = DE_2 + EE_3, &&\nabla_{E_3} E_1 = GE_2 + HE_3, \\ &\nabla_{E_1} E_2 = -BE_1 + CE_2, &&\nabla_{E_2} E_2 = -EE_1 + FE_2, &&\nabla_{E_3} E_2 = -HE_1 + IE_2, \\ &\nabla_{E_1} E_3 = -AE_1 - CE_3, &&\nabla_{E_2} E_3 = -DE_1 - FE_3, &&\nabla_{E_3} E_3 = -GE_1 - IE_3. \end{aligned} \tag{7}$$

Twice contracting the second Bianchi identity, and taking into account that the scalar curvature τ is constant along M, we find that

$$\nabla_{E_1}\rho(E_i, E_1) + \nabla_{E_2}\rho(E_i, E_3) + \nabla_{E_3}\rho(E_i, E_2) = 0 \tag{8}$$

for all $i \in \{1, 2, 3\}$, and it then follows immediately from (7), (6) and (8) that

$$E = 0, \qquad B = 2F. \tag{9}$$

Remark 1. It is easily seen that conditions (5) and (6) only fix the null frame field $\{E_1, E_2, E_3\}$ up to a "null" transformation

$$E_1' = \epsilon_2 E_1 + \alpha E_2, \quad E_2' = \epsilon_1 E_2, \quad E_3' = \epsilon_1\left(E_3 - \epsilon_2 \alpha E_1 - \tfrac{1}{2}\alpha^2 E_2\right), \tag{10}$$

where $\epsilon_i = \pm 1$, $i = 1, 2$, and where α is an arbitrary real-valued function on M, and a straightforward computation shows that such a transformation changes the connection components given in (7) as follows:

$$\begin{aligned} F' &= \epsilon_1 F, \qquad D' = \epsilon_2 D + E_2(\alpha) + \alpha F, \\ C' &= \epsilon_2 C + 3\alpha F, \qquad H' = \epsilon_2 H - 2\alpha F, \\ I' &= \epsilon_1(I + \epsilon_2\alpha(H - C) - \tfrac{5}{2}\alpha^2 F), \\ A' &= \epsilon_1(A + \epsilon_2\alpha(C + D) + \epsilon_2 E_1(\alpha) + \alpha E_2(\alpha) + 2\alpha^2 F), \\ G' &= \epsilon_2 G + \alpha(I - A) + \epsilon_2 \tfrac{1}{2}\alpha^2(H - 2C - D) - \tfrac{3}{2}\alpha^3 F \\ &\quad + E_3(\alpha) - \epsilon_2 \alpha E_1(\alpha) - \tfrac{1}{2}\alpha^2 E_2(\alpha). \end{aligned} \tag{11}$$

It follows from Remark 1 that, choosing the function α to be a solution of the differential equation

$$C - D - E_2(\alpha) + 2\alpha F = 0,$$

we can always specify the null frame field $\{E_1, E_2, E_3\}$ in such a way that

$$D = C. \tag{12}$$

The null frame field is then fixed up to a null transformation (10), where α satisfies the differential equation

$$E_2(\alpha) - 2\alpha F = 0. \tag{13}$$

A straightforward computation using (7), (9), (12) and (6) now yields the following system of differential equations:

$$E_2(F) - 3F^2 = 0, \tag{14}$$
$$E_1(C) + C^2 - E_2(A) + AF + \kappa = 0, \tag{15}$$
$$E_1(H) + H^2 - 2E_3(F) + 2IF + 2AF + \kappa = 0, \tag{16}$$
$$E_1(F) + 4FC - E_2(C) = 0, \tag{17}$$
$$E_2(H) + 2FC - 2FH = 0, \tag{18}$$
$$E_1(G) - E_3(A) + A^2 + 3CG + GH - AI - 2\eta = 0, \tag{19}$$
$$E_1(I) - E_2(G) + FG + CI + HI = 0, \tag{20}$$
$$E_2(G) - E_3(C) + AC - AH + 2FG = 0, \tag{21}$$
$$E_2(I) - E_3(F) + C^2 - 2CH + 2IF - \kappa = 0. \tag{22}$$

Moreover, it is easily seen from (7) that the Lie brackets of the vector fields $\{E_1, E_2, E_3\}$ are given by

$$[E_1, E_2] = -2FE_1, \tag{23}$$
$$[E_1, E_3] = -AE_1 - GE_2 - (C + H)E_3, \tag{24}$$
$$[E_2, E_3] = -(C - H)E_1 - IE_2 - FE_3. \tag{25}$$

Conversely, the Koszul formula [19]

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned}$$

shows that (23)–(25) imply (7), and summarizing the results of this section, we obtain the following.

Theorem 1. *The necessary and sufficient conditions for a Lorentzian manifold* (M, g) *to have a Ricci tensor which satisfies* (6), *is the existence of a null frame field* $\{E_1, E_2, E_3\}$ *and functions* A, C, F, G, H, I *such that the differential equations* (14)–(25) *hold.*

3. Local isometries

The main aim of this paper is to give a complete (local) classification of the three-dimensional Lorentzian manifolds whose Ricci tensor is given by (6). To this purpose, it is important to have a simple criterion to determine if two such manifolds are locally isometric or not. To find such a criterion, let us suppose that (M, g) and (M', g') are two manifolds whose Ricci tensor is given by (6) (with the same values for κ and η), and let $\{E_1, E_2, E_3\}$ and $\{E'_1, E'_2, E'_3\}$ denote the (local) null frame fields along these manifolds constructed as in Section 2. Then we have the following.

Theorem 2. *The differentiable mapping $f : M \to M'$ is a local isometry if and only if*

$$\begin{aligned} f_\star E_1 &= \epsilon_2 E'_1 + \alpha' E'_2, \qquad f_\star E_2 = \epsilon_1 E'_2, \\ f_\star E_3 &= \epsilon_1 (E'_3 - \epsilon_2 \alpha' E'_1 - \tfrac{1}{2}\alpha'^2 E'_2), \end{aligned} \tag{26}$$

where $\epsilon_i = \pm 1$, $i = 1, 2$, and where $\alpha' : M' \to \mathbb{R}$ is a real-valued function such that $E'_2(\alpha') - 2\alpha' F' = 0$.

Proof. If f is a local isometry, it maps the null frame field $\{E_1, E_2, E_3\}$ into the null frame field $\{f_\star E_1, f_\star E_2, f_\star E_3\}$. This frame field satisfies (6),(5) and (12), and (10) and (13) immediately imply the required result.

Conversely, if f satisfies the conditions stated in the theorem, it preserves the null frame field, and hence also the pseudo-orthonormal frame field $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ given by (4), implying that f must be a local isometry. □

Using Remark 1, together with Theorem 2, we can now prove the following result, giving necessary conditions for a three-dimensional Lorentzian manifold whose Ricci tensor satisfies (6) to be locally homogeneous:

Theorem 3. *Let (M, g) be a three-dimensional Lorentzian manifold whose Ricci tensor satisfies (6). If (M, g) is locally homogeneous then*
(1) *$F = 0$;*
(2) *$C = H = \pm\sqrt{-\kappa}$, and hence $\kappa \le 0$;*
(3) *I is constant along M.*

Proof. Let p and p' be two points on M and denote by $\{E_1, E_2, E_3\}$ (resp. $\{E'_1, E'_2, E'_3\}$) the null frame field in a neighbourhood around p (resp. p') constructed as in Section 2. The local homogeneity of (M, g) implies the existence of a (local) isometry $f : M \to M$ such that $f(p) = p'$. It then follows from (26) and (11) that F is constant along M, and (14) implies that $F = 0$. A similar argument shows that H and C are constant along M, and that $E_2(I) = 0$, and it then follows from (16) and (22) that $C = H = \pm\sqrt{-\kappa}$. Finally, the argument used above then yields that I is constant along M. □

4. Local classification

The aim of this section is to make a detailed analysis of the differential equations (14)–(25), and to obtain a complete local classification of the three-dimensional Lorentzian manifolds whose Ricci tensor satisfies (6). To this purpose, we start by determining the solutions of the differential equations, thereby obtaining a family of three-dimensional Lorentzian manifolds whose Ricci tensor is given by (6). Next, we determine which of these manifolds are locally isometric, and combining these results, we obtain a local classification result similar to those in [4,5].

Remark 2. It is easily seen that (14) admits two types of solutions, namely, F is constant (in which case $F = 0$), and F is non-constant, which, by Theorem 2 and Remark 1, lead to two families of solutions that are not locally isometric. In what follows, we will consider these two classes of solutions separately.

Case 1: $F = 0$. In this case, it follows from (23) that we can construct a coordinate system (x, y, z) on a neighbourhood in M such that

$$E_1 = \frac{\partial}{\partial x}, \qquad E_2 = \frac{\partial}{\partial y}, \qquad E_3 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \tag{27}$$

and (24) and (25) then yield that

$$\begin{aligned} \frac{\partial a}{\partial x} &= -A - (C+H)a, & \frac{\partial a}{\partial y} &= -(C-H), \\ \frac{\partial b}{\partial x} &= -G - (C+H)b, & \frac{\partial b}{\partial y} &= -I, \\ \frac{\partial c}{\partial x} &= -(C+H)c, & \frac{\partial c}{\partial y} &= 0. \end{aligned} \tag{28}$$

The complete solution of Eqs. (16) and (18) is given by

$$H = \begin{cases} (x + \bar{H}(z))^{-1} \ \text{ or } \ 0 & \text{if } \kappa = 0, \\ \sqrt{\kappa}\tan(-\sqrt{\kappa}x + \bar{H}(z)) & \text{if } \kappa > 0, \\ \sqrt{-\kappa}\tanh(\sqrt{-\kappa}x + \bar{H}(z)) \ \text{ or } \ \pm\sqrt{-\kappa} & \text{if } \kappa < 0, \end{cases} \tag{29}$$

and we obtain from (28) and (22) that

$$\begin{aligned} c &= \bar{c}(x,z), \quad C = -H - \frac{1}{\bar{c}}\frac{\partial \bar{c}}{\partial x}, \quad a = (H - C)y + \bar{a}(x,z), \\ I &= (-C^2 + 2CH + \kappa)y + \bar{I}(x,z), \\ b &= (C^2 - 2CH - \kappa)\frac{y^2}{2} - \bar{I}y + \bar{b}(x,z), \\ A &= \left(\kappa + C^2 + \frac{\partial C}{\partial x}\right)y - \frac{\partial \bar{a}}{\partial x} - (C+H)\bar{a}, \\ G &= (H - C)\left(C^2 + \kappa + 2\frac{\partial C}{\partial x}\right)\frac{y^2}{2} + \left(\frac{\partial \bar{I}}{\partial x} + (C+H)\bar{I}\right)y \\ &\quad - \frac{\partial \bar{b}}{\partial x} - (C+H)\bar{b}, \end{aligned} \tag{30}$$

while Eqs. (15), (17) and (20) are trivial consequences of (29) and (30). Substituting (30) in the remaining Eqs. (19) and (21), we obtain the following system of two differential equations:

$$
\begin{aligned}
&-(H+C)\bar{I}+\bar{a}\left(C^2-H^2+\frac{\partial C}{\partial x}\right)+\frac{\partial \bar{a}}{\partial x}(C-H)+\bar{c}\frac{\partial C}{\partial z}-\frac{\partial \bar{I}}{\partial x}=0,\\
&\bar{a}^2\left(\kappa-\frac{\partial C}{\partial x}-C^2-2CH\right)\\
&\quad-\bar{a}\left(\bar{c}\frac{\partial (C+H)}{\partial z}+\frac{\partial^2\bar{a}}{\partial x^2}+(C+H)\bar{I}+3(C+H)\frac{\partial \bar{a}}{\partial x}\right)\\
&\quad-\bar{c}\frac{\partial^2\bar{a}}{\partial x\partial z}-\frac{\partial \bar{a}}{\partial x}\bar{I}-\left(\frac{\partial \bar{a}}{\partial x}\right)^2-\bar{c}(C+H)\frac{\partial \bar{a}}{\partial z}+2\bar{b}\left(\frac{\partial C}{\partial x}+2C^2+2CH\right)\\
&\quad+2\frac{\partial \bar{b}}{\partial x}(H+2C)+\frac{\partial^2\bar{b}}{\partial x^2}+2\eta=0.
\end{aligned}
\tag{31}
$$

We then find from (29)–(31) and the Cauchy–Kowalewski theorem that, at least in the analytic case, there exists a family of solutions of (14)–(25) depending on two functions of two variables (namely, $\bar{c}$ and $\bar{a}$) and four functions of one variable (namely, $\bar{H}$ and the initial conditions for $\bar{I}$ and $\bar{b}$).

Now, let (M, g) and (M', g') be two manifolds associated to different solutions of Eqs. (14)–(25), and let (x, y, z) (resp. (x', y', z')) be local coordinates on M (resp. M') constructed as in (27). Then it follows from Theorem 2 that the mapping $f : M \to M'$ given by

$$x' = x'(x, y, z), \qquad y' = y'(x, y, z), \qquad z' = z'(x, y, z)$$

is a local isometry if and only if

$$x' = \epsilon_2 x + \varphi_1(z), \qquad y' = \epsilon_1 y + \mu(x, z) + \varphi_2(z), \qquad z' = \varphi_3(z).$$

We conclude that the family of isometries in this case depends on three functions of one variable and one function of two variables.

Comparing the family of solutions of the differential equations (14)–(25) with the family of isometries for these Lorentzian manifolds, we obtain the following result.

Theorem 4. *The isometry classes of the germs of real analytic Lorentzian metrics whose Ricci curvature tensor satisfies (6) and such that $\nabla_{E_2} E_2 = 0$, are parametrized by one function of two variables and one function of one variable.*

Case 2: $F \neq 0$. In this case, we start by constructing a coordinate system (x', y', z') on a neighbourhood in M such that

$$E_2 = \frac{\partial}{\partial y'}.$$

Then it follows from (14) that

$$F = -\tfrac{1}{3}(y' + \tilde{F}(x', z'))^{-1}$$

and choosing, if necessary, a new coordinate system

$$x = x', \qquad y = y' + \tilde{F}(x', z'), \qquad z = z',$$

we can always assume that

$$F = -\tfrac{1}{3}y^{-1}. \tag{32}$$

Next, writing

$$E_1 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z},$$

it follows from (23) and (32) that

$$a = \bar{a}(x, z)y^{-2/3}, \qquad b = \bar{b}(x, z)y^{-2/3}, \qquad c = \bar{c}(x, z)y^{-2/3},$$

showing that the function b satisfies the differential equation (13). Putting $b = \alpha$ in (10), we then obtain a new null frame field $\{E_1', E_2', E_3'\}$ such that

$$E_1' = \bar{a}y^{-2/3}\frac{\partial}{\partial x} + \bar{c}y^{-2/3}\frac{\partial}{\partial z}, \qquad E_2' = \frac{\partial}{\partial y},$$

and applying a coordinate transformation

$$\tilde{x} = \tilde{x}(x, z), \qquad \tilde{y} = y, \qquad \tilde{z} = \tilde{z}(x, z),$$

where

$$\bar{a}\frac{\partial \tilde{z}}{\partial x} + \bar{c}\frac{\partial \tilde{z}}{\partial z} = 0,$$

we obtain a new coordinate system such that

$$E_1' = \tilde{a}y^{-2/3}\frac{\partial}{\partial x}, \qquad E_2' = \frac{\partial}{\partial y}.$$

Summarizing the above construction, we can always assume the coordinate system (x, y, z) and the frame field $\{E_1, E_2, E_3\}$ to be chosen in such a way that

$$E_1 = \bar{a}(x, z)y^{-2/3}\frac{\partial}{\partial x}, \qquad E_2 = \frac{\partial}{\partial y}, \qquad E_3 = d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f\frac{\partial}{\partial z}, \tag{33}$$

with $\bar{a}f \neq 0$, while F is given by (32).

Integration of Eqs. (17), (18) and (15) yields

$$C = \bar{C}(x, z)y^{-4/3}, \qquad H = \bar{H}(x, z)y^{-2/3} - \bar{C}y^{-4/3},$$

$$A = \bar{A}(x, z)y^{-1/3} - \frac{3}{2}\bar{a}\frac{\partial \bar{C}}{\partial x}y^{-1} - \frac{3}{4}\bar{C}^2y^{-5/3} + \frac{3}{4}\kappa y \tag{34}$$

and from (25) and (33) we find

$$\frac{\partial d}{\partial y} = -(C-H)\bar{a}y^{-2/3} - Fd, \tag{35}$$

$$\frac{\partial e}{\partial y} = -I - Fe, \tag{36}$$

$$\frac{\partial f}{\partial y} = -Ff. \tag{37}$$

Integrating (35) and (37), and taking into account (34), we obtain that

$$d = \bar{d}(x,z)y^{1/3} - \tfrac{3}{2}\bar{a}\bar{H}y^{-1/3} + \tfrac{3}{2}\bar{a}\bar{C}y^{-1}, \qquad f = \bar{f}(x,z)y^{1/3}, \tag{38}$$

while it follows from (36) and (16) that

$$\begin{aligned} e &= \bar{e}(x,z)y^{4/3} + \frac{3}{2}\left(\frac{3}{2}\bar{a}\frac{\partial \bar{H}}{\partial x} - \bar{A} + \frac{3}{2}\bar{H}^2\right)y^{2/3} - \frac{9}{4}\bar{H}\bar{C} + \frac{9}{8}\bar{C}^2 y^{-2/3} - \frac{9}{8}\kappa y^2, \\ I &= -\bar{e}y^{1/3} - \frac{1}{2}\left(\frac{3}{2}\bar{a}\frac{\partial \bar{H}}{\partial x} - \bar{A} + \frac{3}{2}\bar{H}^2\right)y^{-1/3} - \frac{3}{4}\bar{H}\bar{C}y^{-1} + \frac{9}{8}\bar{C}^2 y^{-5/3} + \frac{15}{8}\kappa y. \end{aligned} \tag{39}$$

It is then easily seen that (22) is trivially satisfied. Next, (24) and (33) yield the differential equations

$$\bar{a}y^{-2/3}\frac{\partial d}{\partial x} - d\frac{\partial \bar{a}}{\partial x}y^{-2/3} + \frac{2}{3}e\bar{a}y^{-5/3} - f\frac{\partial \bar{a}}{\partial z}y^{-2/3} = -A\bar{a}y^{-2/3} - (C+H)d, \tag{40}$$

$$\bar{a}y^{-2/3}\frac{\partial e}{\partial x} = -G - (C+H)e, \tag{41}$$

$$\bar{a}y^{-2/3}\frac{\partial f}{\partial x} = -(C+H)f, \tag{42}$$

and (41) immediately yields that

$$G = -y^{-2/3}\left(\bar{a}\frac{\partial e}{\partial x} + \bar{H}e\right), \tag{43}$$

while (40) and (42) lead to the differential equations

$$\bar{H}\bar{f} + \bar{a}\frac{\partial \bar{f}}{\partial x} = 0, \tag{44}$$

$$\bar{a}\frac{\partial \bar{d}}{\partial x} - \bar{d}\frac{\partial \bar{a}}{\partial x} + \frac{2}{3}\bar{a}\bar{e} - \bar{f}\frac{\partial \bar{a}}{\partial z} + \bar{d}\bar{H} = 0. \tag{45}$$

Substituting the expressions found above in (21) we obtain the differential equation

$$\begin{aligned} &2\bar{A}\bar{H} - \frac{9}{2}\bar{a}\bar{H}\frac{\partial \bar{H}}{\partial x} - \frac{3}{2}\bar{H}^3 - \frac{3}{2}\bar{a}\frac{\partial \bar{a}}{\partial x}\frac{\partial \bar{H}}{\partial x} - \frac{3}{2}\bar{a}^2\frac{\partial^2 \bar{H}}{\partial x^2} \\ &\quad + \bar{a}\frac{\partial \bar{A}}{\partial x} + \bar{d}\frac{\partial \bar{C}}{\partial x} - \frac{4}{3}\bar{C}\bar{e} + \bar{f}\frac{\partial \bar{C}}{\partial z} = 0, \end{aligned} \tag{46}$$

while (20) is trivially satisfied. Finally, making use of (44)–(46), it turns out that (19) is equivalent to the differential equation

$$12\bar{f}\frac{\partial \bar{A}}{\partial z} + 12\bar{d}\frac{\partial \bar{A}}{\partial x} + 12\bar{H}^2\bar{e} + 12\bar{a}^2\frac{\partial^2 \bar{e}}{\partial x^2} + 24\eta - 16\bar{A}\bar{e} - 27\kappa\bar{a}\frac{\partial \bar{C}}{\partial x} - 54\kappa\bar{C}\bar{H}$$
$$+ 12\bar{a}\frac{\partial \bar{a}}{\partial x}\frac{\partial \bar{e}}{\partial x} + 12\bar{a}\bar{e}\frac{\partial \bar{H}}{\partial x} + 24\bar{a}\bar{H}\frac{\partial \bar{e}}{\partial x} = 0. \tag{47}$$

Choosing three functions $\bar{a} \neq 0$, $\bar{f} \neq 0$ and $\bar{d}$ of two variables, we can solve (44) and (45) for the functions $\bar{H}$ and $\bar{e}$, and applying the Cauchy–Kowalewski theorem to the differential equations (46) and (47), we find that, at least in the analytic case, there exists a family of solutions to the differential equations (14)–(25) depending on three functions of two variables (namely, $\bar{a}$, $\bar{f}$ and $\bar{d}$) and two functions of one variable (namely, the initial conditions for $\bar{A}$ and $\bar{C}$).

As before, let us suppose that (M, g) and (M', g') are two manifolds associated to different solutions of (14)–(25), and let (x, y, z) (resp. (x', y', z')) be local coordinates on M (resp. M') constructed as in (33). Then it follows immediately from Theorem 2 and our special choice of coordinates that f is a local isometry if and only if

$$x' = x'(x, z), \qquad y' = \epsilon_1 y, \qquad z' = z'(z),$$

showing that the family of isometries in this case depends on one function of one variable and one function of two variables.

Again comparing the family of solutions of the differential equations (14)–(25) with the family of isometries, we obtain the following.

Theorem 5. *The isometry classes of the germs of real analytic Lorentzian metrics whose Ricci curvature tensor satisfies* (6) *and such that* $\nabla_{E_2}E_2 \neq 0$, *are parametrized by two functions of two variables and one function of one variable.*

Remark 3. In [6] the authors proved that a curvature homogeneous three-dimensional Lorentzian manifold modelled on an indecomposable, non-irreducible Lorentzian symmetric space has a Ricci curvature tensor satisfying (6) with $\kappa = 0$, and they constructed a family of non-homogeneous curvature homogeneous examples (depending on one function of one variable), by assuming that the null eigenvector E_2 is recurrent, i.e., $F = H = 0$. The techniques developed in this section allow us to obtain a complete local classification of three-dimensional curvature homogeneous Lorentzian manifolds modelled on an indecomposable, non-irreducible Lorentzian symmetric space, thereby solving (at least in the three-dimensional case) a problem stated in [6]. In the rest of this section, we will use the results stated above to compute a family of solutions similar to the one in [6], as well as some new examples of curvature homogeneous Lorentzian manifolds modelled on an indecomposable, non-irreducible three-dimensional Lorentzian symmetric manifold.

Example 1. We start by constructing a family of explicit examples similar to the one in [6]. To do this, we put

$$F = H = \kappa = a = 0, \qquad c = 1.$$

Then it follows from (30) and (31) that

$$C = 0, \qquad A = 0, \qquad I = \bar{I}(z),$$

and putting

$$b = -\bar{I}(z)y - \eta x^2,$$

we obtain a family of curvature homogeneous Lorentzian manifolds (depending on the function $\bar{I}(z)$) whose Ricci tensor satisfies (6) with $\kappa = 0$. It can be shown using Theorem 2 that these manifolds are pairwise non-isometric and it is easily seen from Theorem 3 that they are, in general, non-homogeneous.

Example 2. To construct a new example, we put $F = \kappa = 0$ and $H = -C = x^{-1}$. Then it follows from (30) and (31) that putting

$$a = 2x^{-1}y, \qquad b = \tfrac{3}{2}x^{-2}y^2 - 2\eta x^2 \ln x, \qquad c = 1,$$

in (27), we obtain a curvature homogeneous three-dimensional Lorentzian manifolds modelled on an indecomposable Lorentzian symmetric space, while (11) and Theorem 2 imply that this example cannot be isometric to the ones given in [6], and Theorem 3 shows that this manifold is not locally homogeneous.

Example 3. Finally, putting

$$F = -\tfrac{1}{3}y^{-1}, \qquad C = H = \kappa = 0, \qquad \bar{a} = 1, \qquad \bar{f} = 1$$

in the computation outlined in Case 2, and choosing

$$\bar{A} = \tfrac{1}{2}\eta, \qquad \bar{e} = 3,$$

we find that the null frame field

$$E_1 = y^{-2/3}\frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial y},$$
$$E_3 = -2xy^{1/3}\frac{\partial}{\partial x} + \left(3y^{4/3} - \frac{3}{4}\eta y^{2/3}\right)\frac{\partial}{\partial y} + y^{1/3}\frac{\partial}{\partial z}$$

determines a curvature homogeneous non-homogeneous Lorentzian manifold modelled on an indecomposable non-irreducible symmetric space, which by Remark 2 is not isometric to any of the examples introduced before.

Remark 4. It follows from Theorem 3 that there exists no homogeneous three-dimensional Lorentzian manifold whose Ricci tensor satisfies (6) with $\kappa > 0$. As an immediate consequence, the curvature homogeneous (non-homogeneous) Lorentzian manifolds associated to $\kappa > 0$ do not admit a homogeneous model space.

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